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LETTER TO THE EDITOR

Quantum Heisenberg group

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Abstract. The Heisenberg algebra $h(3)$ is the Lie algebra of the Lie group $H(3)$ of 3×3 upper triangular matrices with the 1's on the diagonal. This group is quantized. The dual algebra $h(3)^*$ is also quantized, resulting in a quantum coadjoint representation of the quantum group $H(3)_{a,b,c,d,m,n}$. The phase space $T^*H(3)$ is also quantized.

The Heisenberg algebra $h(3)$ is a three-dimensional Lie algebra with a basis e_1, e_2, e_3 and with the commutation relations

$$[e_1, e_2] = e_3 \qquad [e_1, e_3] = [e_2, e_3] = 0\,. \tag{1}$$

From the matrix representation

$$t = \begin{pmatrix} 0 & t_1 & t_3 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

of an element $t = t_1e_1 + t_2e_2 + t_3e_3$ in $h(3)$, we see that the corresponding Lie group $H(3)$ is the group of 3 by 3 uppertriangular matrices with the 1's on the diagonal

$$H(3) = \left\{ g = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \tag{3}$$

Since

$$g^{-1} = \begin{pmatrix} 1 & -x_1 & x_1x_2 - x_3 \\ 0 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

the adjoint action Ad of the Lie group $H(3)$ on its Lie algebra $h(3)$ has the form

$$Ad_g(t) = gtg^{-1} = \begin{pmatrix} 0 & t_1 & t_3 - x_2t_1 + x_1t_2 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

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If A_1, A_2, A_3 are the coordinates on the dual space $h(3)^*$ in a basis dual to the basis e_1, e_2, e_3 of $h(3)$, then the coadjoint representation of the Lie group $H(3)$ on the dual space $h(3)^*$ has the form

$$Ad_g^* = (Ad_{g-1})^* : \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \longmapsto \begin{pmatrix} A_1 + x_2 A_3 \\ A_2 - x_1 A_3 \\ A_3 \end{pmatrix} = \begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \\ \widetilde{A}_3 \end{pmatrix} . \tag{5}$$

The natural linear Poisson brackets on $h(3)^*$ are

$$\{A_1, A_2\} = A_3 \qquad \{A_1, A_3\} = \{A_2, A_3\} = 0 \tag{6}$$

formulae dual to (1). *The coadjoint action (5) preserves these linear Poisson brackets.*

In order to quantize this classical picture, let us consider first the quasi-classical situation. Starting with the group $H(3)$, this means that we have to determine all multiplicative Poisson brackets on it; these define what is called Poisson Lie structures. (The second referee suggests that the reader may need a reference for the standard facts about quantum groups: see, e.g. Drinfel'd 1986.) Ignoring for a moment the question of Jacobi identities, the space of such (pre) Poisson brackets on the group $H(3)$ can be shown to be six dimensional

$$\begin{aligned} \{x_1, x_2\} &= bx_1 + ax_2 \\ \{x_1, x_3\} &= cx_1 + \frac{b}{2}x_1^2 + mx_2 + ax_3 \\ \{x_2, x_3\} &= dx_2 - \frac{a}{2}x_2^2 + nx_1 - bx_3 \end{aligned} \tag{7}$$

where a, b, c, d, m, n are arbitrary constants. The Jacobi identities impose the following two relations among these constants:

$$b(d-c) - 2an = 0 \qquad a(c-d) - 2bm = 0 . \tag{8}$$

Thus, the space of Lie Poisson structures on the group $H(3)$ is four dimensional.

Now, the coadjoint representation (5) of $H(3)$ on $h(3)^*$ is no longer a Poisson action since we have introduced non-zero Poisson brackets (7) on the group $H(3)$ itself. To restore the harmony, the linear Poisson brackets (6) on $h(3)^*$ have to be modified. A straightforward computation shows that the most general Poisson brackets on $h(3)^*$ for which the coadjoint action (5) is Poisson, are of the form

$$\begin{aligned} \{A_1, A_2\} &= A_1\varphi + A_2\psi + \theta \\ \{A_1, A_3\} &= A_3\psi + bA_3^2 \\ \{A_2, A_3\} &= -A_3\varphi + aA_3^2 \end{aligned} \tag{9}$$

where φ, ψ, θ are arbitrary functions of A_3. The Jacobi identity among A_1, A_2, A_3 imposes the relation

$$(\psi + bA_3)\frac{d\varphi}{dA_3} = (\varphi - aA_3)\frac{d\psi}{dA_3} . \tag{10}$$

If we require the invariant A_3 of the coadjoint action to belong to the Poisson centre of $h(3)^*$, we get

$$\varphi = aA_3 \qquad \psi = -bA_3$$

so that the relation (10) is satisfied and formulae (9) take the final form

$$\{A_1, A_2\} = A_3(aA_1 - bA_2) + \theta(A_3) \qquad \{A_1, A_3\} = \{A_2, A_3\} = 0\,. \tag{11}$$

This is our quasi-classical picture. As a quantum analogue of the quasi-classical formulae (7), (11) we set

$$\begin{aligned} [x_1, x_2] &= bx_1 + ax_2 \\ [x_2, x_3] &= dx_2 - \frac{a}{2}x_2^2 + nx_1 - bx_3 \\ [x_1, x_3] &= cx_1 + \frac{b}{2}x_1^2 + mx_2 + ax_3 \end{aligned} \tag{12}$$

and

$$[A_1, A_2] = A_3(aA_1 - bA_2) + \theta(A_3) \qquad [A_1, A_3] = [A_2, A_3] = 0 \tag{13}$$

where a, b, c, d, m, n are arbitrary constants, and $\theta(A_3)$ is an arbitrary function (e.g., $\theta(A_3) = A_3$). Formulae (12), (13) have the following easily verifiable properties:
(A) Formulae (12) are *multiplicative*. This means that if

$$\overline{g} = \begin{pmatrix} 1 & \overline{x}_1 & \overline{x}_3 \\ 0 & 1 & \overline{x}_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and if $[x_i, \overline{x}_j] = 0, \quad 1 \leqslant i,\ j \leqslant 3$, then the parameters X_1, X_2, X_3 of the product

$$g\overline{g} = \begin{pmatrix} 1 & X_1 & X_3 \\ 0 & 1 & X_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + \overline{x}_1 & x_3 + \overline{x}_3 + x_1\overline{x}_2 \\ 0 & 1 & x_2 + \overline{x}_2 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfy again the defining commutation relations (12);
(B) The deformation (12) is flat (i.e., it satisfies the PBW property) iff (cf (8))

$$b(d - c) - 2an = 0 \qquad a(c - d) - 2bm = 0 \tag{14}$$

(C) The coadjoint action (5) preserves the commutation relations (13);
(D) The inverse map i

$$i(g) = g^{-1}$$

changes the sign of all the constants a, b, c, d, m, n in (12). In the quasi-classical situation (7), this map is anti-Poisson.

Remarks. (i) For the Lie groups $H(n)$ of n by n upper triangular matrices with the 1's on the diagonal , the Poisson and quantum formulae become increasingly messy for $n > 3$.

(ii) In classical mechanics, on the phase space T^*G of the configuration space of a Lie group G, the canonical symplectic from $\mathrm{d}p \wedge \mathrm{d}q$ is invariant with respect to the lifted to T^*G both left *and* right actions of G on G. In the quantum case one has *two different* pictures, one each for the left and the right actions of G on itself. For example, for the case at hand, with G being $H(3)$, the left action of $H(3)$ on itself has the form

$$L_x : \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \longmapsto \begin{pmatrix} q_1 + x_1 \\ q_2 + x_2 \\ q_3 + x_1 q_2 + x_3 \end{pmatrix} \qquad \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \longmapsto \begin{pmatrix} p_1 \\ p_2 - x_1 p_3 \\ p_3 \end{pmatrix}. \tag{15}$$

The form $\boldsymbol{p}\mathrm{d}\boldsymbol{q}$ is preserved under this action, for each fixed $\boldsymbol{x} \in H(3)$. Turning on the Poisson brackets (7) on $H(3)$ necessitates a modification in the canonical Poisson bracket $\{p_i, q_j\} = \delta_{ij}$ on $T^*H(3)$. There are *many* such modifications (cf formulae (9); that is why we avoid the R-matrix language). The simplest one has the form

$$\{q_i, q_j\} \text{ as in (7)} \qquad \{p_1, q_i\} = \delta_{1i} \qquad \{p_3, q_i\} = \delta_{3i} \qquad \{p_2, q_1\} = 0$$
$$\{p_2, q_2\} = 1 - p_3(bq_1 + aq_2) \qquad \{p_2, q_3\} = -p_3\left(aq_3 + \frac{b}{2}q_1^2 + cq_1 + mq_2\right) \tag{16}$$
$$\{p_i, p_j\} = 0 .$$

The action (15) is now a Poisson map

$$H(3) \times T^*H(3) \longrightarrow T^*H(3) .$$

Formulae (16) can be quantized, as follows

$$[q_i, q_j] \text{ as in (12)} \qquad [p_1, q_i] = \delta_{1i}h \qquad [p_3, q_i] = \delta_{3i}h \qquad [p_2, q_1] = 0$$
$$[p_2, q_2] = h - p_3(bq_1 + aq_2) \qquad [p_2, q_3] = -p_3\left(aq_3 + \frac{b}{2}q_1^2 + cq_1 + mq_2\right) \tag{17}$$
$$[p_i, p_j] = 0 .$$

A direct check shows that the commutation relations (17) are preserved under the left action (15). For the right action, we have the following analogues of formulae (15)–(17):

$$R_x : \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \longmapsto \begin{pmatrix} q_1 + x_1 \\ q_2 + x_2 \\ q_3 + x_2 q_1 + x_3 \end{pmatrix} \qquad \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \longmapsto \begin{pmatrix} p_1 - x_2 p_3 \\ p_2 \\ p_3 \end{pmatrix} \tag{18}$$

$$\{q_i, q_j\} \text{ as in (7)} \qquad \{p_2, q_i\} = \delta_{2i} \qquad \{p_3, q_i\} = \delta_{3i} \qquad \{p_1, q_2\} = 0$$
$$\{p_1, q_1\} = 1 + p_3(bq_1 + aq_2) \qquad \{p_1, q_3\} = p_3\left(bq_3 + \frac{a}{2}q_2^2 - dq_2 - nq_1\right) \tag{19}$$
$$\{p_i, p_j\} = 0$$

$$[q_i, q_j] \text{ as in (12)} \qquad [p_2, q_i] = h\delta_{2i} \qquad [p_3, q_i] = h\delta_{3i} \qquad [p_1, q_2] = 0$$
$$[p_1, q_1] = h + p_3(bq_1 + aq_2) \qquad [p_1, q_3] = p_3\left(bq_3 + \frac{a}{2}q_2^2 - dq_2 - nq_1\right) \tag{20}$$
$$[p_i, p_j] = 0.$$

We see that formulae involving the left action, the right action, and the coadjoint action, are all different.

Remark. The r-matrix language has been avoided so far, in recognition of the minor role r-matrix notions play for the Heisenberg group: if $r = \alpha e_1 \wedge e_2 + \beta e_1 \wedge e_3 + \gamma e_2 \wedge e_3$ is an element of $h(3) \wedge h(3)$ then r automatically satisfies the GYBE (generalized Yang–Baxter equation), while r is the classical r-matrix iff $\alpha = 0$. Such a r-matrix leads to the case

$$a = b = m = n = 0 \qquad c = d = \alpha \tag{21}$$

of the Poisson brackets (7), which is only a one-dimensional line in a four-dimensional space. Notice that classical r-matrices yield zero Poisson brackets on the group. Similarly, the commutation relations (12) can be derived from the R-matrix ansatz

$$Rg_1g_2 = g_2g_1R \qquad g_1 = g \otimes \mathbf{1} \qquad g_2 = \mathbf{1} \otimes g \tag{22}$$

only for the case (21), when

$$R = \mathbf{1} \otimes \mathbf{1} + c(E_{12} \otimes E_{23} - E_{23} \otimes E_{12}) \,. \tag{23}$$

(The first referee points out the paper Celeghini *et al* (1991) where this one-dimensional part of the full quantum relations is found.)

References

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